WHAT SENSE DO CHILDREN MAKE OF WHAT THEIR MATHEMATICS TEACHERS SAY AND DO?

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Abstract

Teachers use a variety of means to communicate mathematical concepts and procedures to children. They use language and other symbols along with physical objects to provide children with opportunities to construct meaning, but what sense do the children make of their teacher's words and actions? The study reported in this paper was a pilot for a longitudinal study to investigate the development of young children's mental representations that are formed as a result of their interaction with their teacher's external representations. The results suggest that children's mental representations, for the decimal numeral system and multi-digit calculation, show influences of the external representations. Furthermore that children express their methods of calculation in metaphoric language which indicates the re-presented experience.

Περίληψη

Οι δάσκαλοι χρησιμοποιούν διάφορα μέσα για μεταδώσουν τις μαθηματικές έννοιες και διαδικασίες στους μαθητές. Χρησιμοποιούν γλωσσικά και άλλα σύμβολα, όπως και φυσικά αντικείμενα, προκειμένου να προσφέρουν στους μαθητές την δυνατότητα να σχηματίσουν νοηματικά θεμέλια, αλλά το ερώτημα είναι: τί καταλαβαίνουν οι μαθητές από τα λόγια και τις πράξεις των δασκάλων τους; Η μελέτη που αναφέρεται σε αυτό το άρθρο ήταν μια πιλοτική έρευνα με σκοπό να εξετάσει την εξέλιξη των νοητικών αναπαραστάσεων των μικρών παιδιών, οι οποίες σχηματίζονται ως αποτέλεσμα των αλληλεπιδράσεων με τις εξωτερικές αναπαραστάσεις που τους προσφέρει ο δάσκαλος τους. Τα αποτελέσματα υποδεικνύουν οτι οι νοητικές αναπαραστάσεις των μαθητών, για το δεκαδικό σύστημα και τις πράξεις με αριθμούς πολλαπλών ψηφίων, επηρρεάζονται από τις εξωτερικές αναπαραστάσεις των δασκάλων τους. Επιπλέον, οι μαθητές εκφράζουν τις μεθόδους υπολογισμών τους χρησιμοποιώντας μεταφορική γλώσσα η οποία υποδηλώνει την ανα-παράσταση των εμπειριών τους.

Introduction

As a working definition, an external 'representation' of a mathematical concept or process is taken to be any linguistic or physical devise (written or spoken, words or symbols, pictures or concrete objects) which 'stands-in' for, illustrates or exemplifies that concept or process. Thus teachers' representations are the means by which teachers attempt to communicate mathematical ideas. Whatever comes to a child's mind, when asked to recall a concept or perform a procedure that the teacher has attempted to communicate, is referred to as their 'mental representation'. This does not imply that the mental representation is a representation, in the sense of a copy, **of** the teacher's representation, but that the child's mental representation is formed **from** experience of the teacher's representation. Like the teachers' representations the children's mental representations 'stand-in' for their experiences of, and may thus represent for them, the concepts and processes that were intended to be communicated. These mental representations may be quasi-sensory 'images' that have perceptual qualities (visual, verbal or tactile) that capture surface characteristics of a particular teachers' representation. Some children may, however, construct a mental representation that embraces the concept or process that is common to a variety of teachers' representations. My conjecture is that, whilst the children's internal representations are their own constructions, their mental representations reproduce aspects of one or more of their teachers' representations. Furthermore it is possible that the mental representations that children form from these early experiences may influence the way in which they learn future concepts and procedures in mathematics.

Previous exploratory studies with children aged 6 to 7 years in an English school have revealed that children seldom used mental visual images when asked to perform calculations without the aid of paper or physical equipment (Bills, in press) . The children did, however, form mental representations which were 'image-like' in reproducing aspects of their teacher's representations (Bills and Gray, 1999) and they used metaphoric language which suggested that the origins of their conceptualisations were rooted in the classroom experiences with the teacher's representations (Bills, 1999). The results reported in this paper come from an interview conducted with children in another class in the same school for the purpose of selecting a second sample for comparison in a future longitudinal study. The findings from the previous study were confirmed but there was also evidence that children who used different categories of mental representations, for mental calculation, gave qualitatively different answers in response to the question “What is the first thing that comes into your head when I say ‘Hundreds tens and units’?”. There was also evidence of a similar style of response to this and a non-arithmetical questions of the same type and similar styles of answers were given to “Tell me how to write the date” and “Tell me how to add 20”. This suggests that the style of mental representation is characteristic of the individual learner, not the context.

This paper provides a brief review of some of the literature which has influenced and informed the study and gives some details of the phenomenographic methodology. The results are presented to illustrate the different categories of mental representation used and how they were identified through an analysis of the children’s language. Linguistic pointers to the children’s facility with generalisation and illustrative examples are also identified. Implications for a longitudinal study are drawn.

Representations and Re-presentations.

The types of representations that are available to teachers have been identified as: real world contexts, manipulable models, pictorial, spoken language and written symbols (Lesh, Post, and Behr, 1987). The teacher in this study used representations of each of these types: money, Deines blocks, hundred squares, verbalised procedures (for example “add the tens and add the units”) and written algorithms. As noted in the introduction a representation is used to communicate a mathematical idea. The representations that teachers use are not *the mathematics* but a transformation of the mathematics into a communicable form (Kang and Kilpatrick, 1992). It is also important to note that a teacher’s presentation of an idea may include a variety of types of representations. One teacher’s words and gestures whilst using a model, for instance, may be quite different from another teacher so that ‘the teacher’s representation’ is peculiar to a particular teacher not just to the material in use.

Concrete-material representations such as Dienes blocks, hundred squares and number tracks, used for place value, are intended to be ‘structure-oriented’ (Resnick and Ford, 1981) but the ‘transparency’ (Meira, 1998) of the correspondence between the material and the structure is variable. Whilst the structure is apparent to the teacher, who already knows the mathematics, it may not be apparent to the child who is supposed to learn it. Bereiter (1985) has termed this the “learning paradox” because children are expected to discern the mathematical meaning represented by the material before they have constructed the meaning; the concepts supposedly ‘embodied’ by the material are more complex than those the children have available. Hall (1991) proposes a ‘Procedural Analogy Theory’ to evaluate the learning potential of representations. This assumes that the value of a physical embodiment lies in the degree of similarity between the embodiment procedure and the procedure for a symbolic representation.

In ‘Metaphors We Live By’, Lakoff & Johnson (1980) have suggested that the use of metaphor indicates *and shapes* our conceptualisations of the real world. Our use of the language of one concept to communicate another indicates that one concept has shaped our understanding of the other. In the context of this paper, the use, by children, of the language associated with a teacher’s representation is taken to indicate ‘the metaphor they calculate by’. Rowland (in press) notes that the use of “you” to refer to generalities is common in non-mathematical situations where “you” is used in place of the more formal “one”, particularly by children, for instance in their description of rules of games. He also suggests, however that the use of the pronoun “you” is an effective ‘pointer’ to a quality of thinking. For the child in his study the shift from “I” to “you” in a problem solving setting signified her move from working with specific numbers to expressing a generalisation.

Pimm (1995) sees ‘manipulation’ as the core metaphor for doing mathematics. He also notes the common belief that mathematical concepts can be more easily understood if represented by physical objects. He suggests, however, that children may end up merely manipulating these physical objects so that they have the experience but miss the meaning. He also suggests that Deines’ material provides physical symbols of the numeral system and are manipulated to mimic the manipulation of the written system. Thus, in his opinion, Dienes blocks symbolise the way the notation system works, not the other way round. The rods are used to model the manipulation of symbols which are themselves substitutes for numbers. Hughes (1986), however, concluded that written procedures and manipulation of materials were seen as fundamentally unrelated activities by the children he studied.

Janvier (1987) notes that the ‘multiple-embodiment approach’ adopted by many teachers follows Dienes’ principle that an ideal method of learning mathematics would be to use several representations of the same object. Dufour-Janvier, Bednarz and Belanger (1987, p.111), however, raise questions about the value of children being presented with a variety of representations, noting that it is often done in the “hope” that children will be able to “... grasp the common properties of these diverse representations and ultimately ‘extract’ the intended structure” or in the hope that children will “... be able to find one which will help”.

The way in which children form mental representations from their mathematical experiences is open to debate. Olson and Campbell (1993) suggest that the Piagetian view of the way mental representations are constructed by the individual stands in the middle ground between theorists such as Fodor, who assume an innate representational language of thought and others, such as Vygotsky, who insist on the priority of public external representations in the development of the individual's internal representations. Fodor argued that it is a particular property of the mind to represent the world in terms of this representational language. Vygotsky adopted Durkheim's argument that logical life has its first source in society so that external representations, including language, are internalised as instruments of thought.

The study I report in this paper takes the constructivist view, as expressed by Lesh and Kelly (1997):

Humans interpret their experiences using internal conceptual structures, which cannot merely be received from others, but which must be developed, actively, by each individual. Further we assume that the meanings of these constructions tend to be partly embedded in a variety of external systems of representation. (p.398)

My conjecture that the child's mental representation 'reproduces' aspects of teachers' representations does not imply that the one is a copy of the other. Von Glasersfeld (1995) suggests that:

We can not share our experience with others, we can only tell them about it, but in doing so, we use the words we have associated with it. What others understand when we speak or write is necessarily in terms of the meanings their experience has led them to associate with the sound images of the particular words - and their experience is never identical with ours. (p.48)

Furthermore, he suggests that to understand something we need to 're-present' an experience that fits with it and that 're-presentation' is a reconstruction, from our memory, of experiences of the world, not a picture of the real world. In the classroom, children interact with their teacher's representations and learn to attach the teacher's words to their experience of those representations. This is not an empiricist view that knowledge results from passive reception of perceptual data because it assumes that the children build concepts from their experiences. These experiences are, however, oriented by teachers through their use of representations.

Method

The research approach adopted is a naturalistic qualitative one which can be termed 'phenomenographic' (Marton, 1988). A phenomenographic study is an investigation of people's understanding of phenomena which seeks to categorise and explain the qualitatively different ways in which people think about the phenomena. The assumption made for my research is that the mental representations formed by children can not be studied in isolation from the context of the classroom and the interaction between the children and teachers. It is essential to observe the common experiences of the children as a basis for the analysis of their different conceptualisations.

The preliminary studies were conducted in a Primary school in a large middle-income village near Birmingham, England, which has children in three sets, by ability, for Mathematics in each year group. In the period October 1997 to July 1998 one lesson was observed each week with Set 1 in Year 2 (aged 6 and 7 years) and individual interviews audio taped each term with a sample of children. From September 1998 these same children, then in Year 3 Set 1 (aged 7 and 8 years) with a new teacher, and also Year 3 Set 2, were observed and interviewed. Results presented in this paper are taken from the interview conducted with twenty children from Year 3 Set 2 in October 1998 for the purpose of selecting the sample for a longitudinal study. The interview consisted of eight questions presented verbally. The first four questions required a calculation to be made without the aid of paper or physical objects and after each the children were asked "What was in your head when you were thinking of that?" These were followed by a pair of questions "Tell me how to ..." and a pair of questions "What is the first thing that comes into your head when I say '...'?". In transcripts a '.' indicates a five second pause and ',' a short pause.

Results

The teachers' representations

The teachers in this school used several representations for the numeral system and to demonstrate two-digit addition and subtraction to Year 2 classes during the course of the year. These included: a number track (from 1 to 105), a hundred square (from 1 to 100), Dienes base ten blocks, numeral cards printed with single digits, coins, chanting number sequences and the written algorithm. The children practised a representation-specific procedure with each of the materials. They added tens, for instance, by: taking ten steps on a number track, going down a column on a number square, having an extra ten-block, replacing the tens digit with one higher, having an extra 10p coin, saying the next word in the word sequence and adding one to the tens column.

The following is taken from a typical lesson in Year 2 in which Dienes blocks were used:

The teacher gave Mandy two 'tens' and four 'ones'. When asked "How many altogether?" Mandy said "Twenty-four". Nina was given one 'ten' and two 'ones' and she responded correctly to a similar question. The teacher requested Mandy and Nina to "put them together in my hands". In response the two children put the 'tens' in one hand and the 'ones' in the other. The teacher then asked "How many altogether?" adding "Look how easy it is to add them instead of all individual cubes".

The handling of 'the tens' and 'the units' separately provides a metaphor for calculations. Individual digit manipulation was also a common feature of Year 2, as another lesson later in the year illustrates:

“You looked at rolls of raffle tickets like this… (| 186 | Drawn on board) What comes next? Why wasn’t the 1 or the 8 changed?” A child replies “Because you are not adding tens or hundreds.”

The teacher next wrote 199 in the middle position and again asked “What comes before? What comes after?” Going on to comment: “But that means I'm altering the tens and hundreds. That’s because I can’t have more than 9 in any column.”

To illustrate “going to” the next number she held up three numeral cards and then changed the units digit card for a different one. She indicated that only the units digit changed except when the 9 is changed for a 0 and then the ten digit is changed as well.

‘Counters’ and ‘manipulators’

In the interviews with children the language of handling the tens or units and of changing digits reveals the influences on their mental representation.

The responses to the question “What comes next after 379”, fell broadly into two categories:

‘counting/adding’:

Joe . 380. well I thought um I knew that, that, I just went one over

Shaun three hundred and, 80. . One more added on to it

Bobby 380 Because um it's 379 and add one

Steve 380 Because you just add one more

Paddy three hundred and, eighty I thought, so if I count from 79, I just have to count one, so I'll just count one and see what, see what it come up to.

Elaine three hundred and, eighty three hundred and seventy-nine, and if you just add one more on it comes to 380

and ‘digit manipulation’:

Rose . three hundred and, eighty. Because take the three hundred off and um just, um, um, just turn the um, seven into an 8

Myles three hundred and *eighty*. Well you know after nine it comes on to a different number

Digit manipulators, however, most frequently gave the wrong answer:

Nikki . three hundred and *eighty* nine Well if you go 379, in the middle it's a 7, so when you go 6, 7 then if it's 8 so you go "right that's 8 then"

Suzy ... 400 cause, after 3 comes 4?

Geoff 4 hundred and, seventyyyy, . ten?

Dave. three, hundred, and ten? Well I thought it went 9, 10 , so it's 10

Some children separated the three hundred then counted:

Pam .. 380 .. well, . well I knew that after 70, 9, eighty came, and, 300 would come first

Nora . 380 well I thought when you go 1, 2, 3, 4, like that I thought that and then I went to 79 and like 80 and I just added a 3 in front

Jeremy um . that's 400, .. um, . 300 and, *eighty*-one well, I thought of the hundreds, what number it was, and then I just counted on with the numbers, counting in ones.

These nine children, who’s mental representation for increasing a number by one involves separating the digits, seem to have been most influenced by the teacher’s representations. Five of them were unsuccessful either because of increasing the wrong digit or, perhaps, when concentrating on a single digit the rest of the number was forgotten.

One child could not answer and one replied 378 but gave no explanation. The other two children seemed to have been influenced by their experience with other teachers’ representations:

Sean I was thinking about how many pennies were in that, how, how I worked it was, I just worked it out with my brain. 'cause in your maths you like, just pound coins and then we added up the number easily.

Robert Well you um you get 370, in your head, and then you like think is it lower, is it low or is it high, and I thought, it's 370 so you had to go three hundred and eighty. Like a square and it's (379), it's in the middle.

Questioning revealed that Robert seemed to be re-presenting his experience of the teacher’s representation concerning rounding-to-the-nearest ten. The teacher had drawn sections of a number track and wrote numbers from, for instance, 370 to 380 in separate squares to show 379 was closer to 380 than 370.

The metaphor of manipulation

Children again divided themselves into ‘counters’ and ‘digit manipulators’ in response to “What is 247 add 10?” Three children said that they could not do this question and five either gave up or gave the wrong answer after attempting to count-on from 247. Counting-on from 247 was only completed successfully by one child. Another child, however counted-on from 47, using her fingers, after separating the two hundred:

Nora .... um 257. well got um 47 was there so I just cut off the 2 and I just left that alone then I got my fingers and I went like that.

Nora’s use of “47 was there”, “cut off” and “left that alone” illustrate the way in which children's mental representations use metaphors of position and handling. This is seen most obviously in the responses of the following digit-manipulators:

Paddy two hundred and *fifty*seven Because um I done it in my head this time because um, 'cause I said, because **I don't need to do anything with the 2** I'll just do that, and then I said **I need to do something with the tens** but **I don't need to do anything with the 7** so I just said if it's going to be 10 well **the seven's going to stay** and after 40, 50 so it's going to be 57 and **then just put 2 back on**.

Sean .. Um 2 hundred and *fifty*seven I was thinking about if you **added another ten**,

Again digit manipulation was frequently wrong:

Hester , 3 hundred, and 47 Because you just **add another ten on** and, you, add, what was the last bit? So **I added on the ten** and I **put 47 on the end**.

Hester’s mental representation had formed, like Paddy and Sean, as a result of experiences of working with separate digits. It was the tens digit that she was attempting to add another ten on to but unfortunately it was the hundreds digit she manipulated. Another child did the same but her language was different:

Nikki , is it *three* hundred and 47? I thought if you, well when it's 10 add 10 it's 20, you just **change the one into a 2** and then you just do the same again and I got the answer.

Other unsuccessful manipulation led three children to give answers of 547, 162 and 550. Only one child had a very different mental representation:

Shaun .. two hundred, and fifty, sssseven. Well it was like moving the numbers along. Thinking about written down numbers.

Subsequent questioning failed to reveal quite what was in Shaun’s mind but it could have been influenced by number tracks.

These children were not very confident with numbers over one hundred because the majority of their mathematics time in Year 2 had been devoted to representations of numbers less than one hundred. Given the question “What is 73 take 20?” many more children were successful with digit manipulation:

Paddy . 53 I was doing it with my fingers because I went like that, 7, 7 take away 2 would be 5 and then I'll just **add the other number on.**

Elaine ..... is it 53? 'cause I was just, I **keep the 3 on another side** and then, because we had to take 20 off and then 70, saying it in my head, so I counted 70 back in tens, like um, it's 70, 60, 50 and then I **put the 3 back on** and it's 53.

Paddy’s response uses the common classroom language of “adding” the 3 to the 5 to give 53. The teacher and children use “add” in the broad sense of ‘putting with’ so they often talk of “taking off” the units digit, doing some manipulation and then “adding” it back on. Paddy made no mention of 50 and there was no sense of him counting-on three to add the number on. Elaine did count back in tens but again “put the 3 back on” rather than counting. Nora explicitly mentions the zero of the 70 but manipulates just the 7:

Nora .... 63. Well I thought, I just **left the zero** and like and thought 7 take away 2 is, oh got it wrong, 5 **add a 3 on** is 53.

Six other children successfully manipulated separate digits and only two were unsuccessful when attempting to manipulate, giving answers of 43 and 7. This higher success may be attributed to the fact that two-digit numbers had been more frequently manipulated both mentally and on paper. Part of the difficulty with manipulating three-digit numbers as separate digits is that a digit may be forgotten or confused with another.

Four children were unable to give an answer to “What is 247 add 10?”, two got lost when attempting to count back in ones and one gave the answer 55 after counting. The other two had different mental representations. Shaun again talked of the number moving, backwards this time, and gave the correct answer. Bobby appeared to have simply manipulated the digits but also put in a remark that is a common classroom metaphor:

Bobby .. 53. Because if you **add** something it **will make it bigger**, but subtracting, **take away** something it **makes it smaller**. Because it was taking away 2 tens.

I refer to this as a metaphor because it uses the language of supplementing or diminishing a collection of countable objects or of increasing or decreasing a quantity such as length. When the teacher and children use this metaphoric language the only things they actually manipulate are numerals.

Representation-specific procedures

Children indicate the teacher’s representation that has influenced them by using metaphoric language and they indicate that the representation-specific procedures are part of their mental representation by using classroom language associated with it. They also signal their use of the teacher’s procedure by the use of “you” and there is further evidence of a recognition of the generality of a procedure by the use of “if”. Notice above that Bobby said “if you add”. This was not an isolated incident. In many of the transcripts quoted so far in this paper “you” appears when children are talking in general terms even though it was in the context of “What was in your head when you were thinking of that?”. For example:

Suzy Well the 4 co, the 5 comes after the 4 and then **you** just add the ten on,

Steve Because I know my ten times table, and **you** just add ten more and **you** get the answer

Geoff .... *fifty*three I thought of it in my head. I like said to myself, like we have 7 and **you** take away 2 and then it's 5 and then 50 and then I just got the 3 on.

Geoff gives an even bigger impression of a re-presented experience by his use of “I said to myself” prior to indicating that “you take away 2” in order to take away 20. This strong sense that the problem is solved by re-presenting in general form a teachers representation-specific procedure was also apparent in one of Sean’s responses:

Sean **I was thinking about if you added another ten, it would make, if you**, I'd have said 7 add 10, that would be 17, so if I said 147 it would be one hundred and *fifty*seven

Sean seems to call up a mental representation of a general procedure, illustrate it with a different example then use it for the problem given (though in doing this he has forgotten that it was 247 ,not 147, add 10).

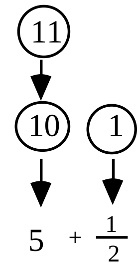
Re-presenting experience

The similarities between the children’s mental representations and the teacher’s representation was also apparent when the children were asked about a procedure that had only been used in just two lessons one week before the interview. There seemed evidence here that the mental representation was a re-presentation of the original experience.

The lesson I observed followed on from the previous day when the idea of “splitting a number in two” was introduced and a method shown for halving an odd number. MP was the teacher and there was a good illustration of the classroom use of “you”. As the child talked the teacher drew a diagram on the white board:

MP How to half an odd number equally? Do you remember what to do?
P 11 is made out of 10 and 1 and you put ...
MP Carry on “you put ...”
P You put half of 10 is 5 and half of 1 is half so it's 5 and a half.

On Board



In the interviews half the children could not answer the question “What is 9 split in two?”. Two said “4 and 5”. The rest gave responses which quite accurately re-presented this experience when asked “What was in your head when you were thinking of that?”. For example:

Dave..... not sure (He was then reminded that there had been a lesson the previous week) It's like when she splits in half and writes the number in the halves. Well **you** put a kind of circle, a number in a circle and then another number that it can be split into half, then I , then **you** put the number in the half.

Jeremy ...um .... um well, **you**, **you** get um 9 like **you** put 1, um, and 9 and then **you** put a little arrow down and the one's like a half and um, **you** add it up, and um the number nine **you** put nine and a half?

Nora Well **you**, **you** get 9 is made out of 8 add 1 and half of 8 , is, 4, and half of 1 is half, um so **you** add 8 and a half together and **you** get 8 and a half, 4 and a half, 8 and a half.

Bobby .. well **you** say half of them, .. it's made up of 8 and one, . um, then **you**, . um say what half of 8 is, and then um, . **you**, ... well **you** can um, . um say that um what half the 9 is 'cause **you**'ve already got half of 8 so, **you** just need to, add um, half of 9 ... four and um, half

Only three of the six gave an unequivocal correct answer. Two other children gave the correct answer. One of these said it “just came into my head” but the other had a mental representation of halving that was not so easily attributable to this particular teacher’s representation.

Consistency in mental representation

In the above examples the children spontaneously gave these descriptions of procedures in response to “What was in your head when you were thinking of that?”. In order to explore children’s ability to express general procedures, without a specific calculation performed first, two questions were included for this purpose “Tell me how to add twenty on to any number” and “Tell me how to write the date”. Only two of the twenty children interviewed gave a procedure in general terms but they did so for both questions:

Shaun	**You** just add a ten and then again	. Just like get what the day it is and then put the month and then the years
Bobby	.. um **you** count how many, ones, um units, to make um the sum and then **you** add the rest on	. **you** put, the, . what day it is, the month it is and the year it is.

Six of the twenty gave an illustrative example for each, indicated by the use of ‘if’ and ‘like’. For example:

Jonathan	. well **if** there was **like** 47 or something **like** that then **you** knew it would be sixty something so **you** would just go 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, on **you**r fingers and do it again	**if** it was **like** September the 6th I'd write, and um 98, **You**'d write 6 dot ,nine nineteen ninety eight
Geoff	. um, . **you**'ll **like** have 20, then **you** have 5 there and then **you** just like 20 and then **you** add on like 5 **if you** were ...	um, . well **you** write, well **if** it's the 20th of October then **you** write 20 and then t, h and then um, . c, t, o, . o, b, e,r

Steve	Well **if you**'re going to add up to twenty, uh, 60, **you** add the tens like twenty, (mutters) 4 more tens, **you**'ve only got 2 and **you** had 4 more tens, **you** have 60	Well in maths **you** always write the short date, **like** today's date 20 dot 10 dot 98

The children were also asked "What is the first thing that comes into your head when I say 'hundreds tens and units'?" and "What is the first thing that comes into your head when I say 'sentence'?" All but three of them were consistent in their type of mental representation. The majority responded with a 'particular' example of each. For instance:

Pam	.... 2 hundred, and sixty, seven	I jumped on a cat

Three talked in more 'general' terms:

Bobby	..... I know how many um ones in tens, in hundreds	um, **you** have to have a doing word in a sentence, a verb, um, and **you** have to have, um, a capital letter at the beginning of a sentence and a full stop at the end.
Joe	Well just put, the hundreds, but it has to be over a hundred or (incomprehensible) and then **you** have to do tens, and then **you** just put the um ones next.	well, it's, **you** just write down um, and **you**, write lots of words, until like **you** get to a place where **you** can put a full stop, to make a sentence
Hester	. 5 hundred and 3, and I know ones really high and ones really low.	**You** write words in sentences.

Some gave responses which use an 'illustrative example'. For instance:

Steve	. **you** um, first thing that came in my head was **you**'ve got three boxes, and **if you** have to copy out a number **like** 937, **you** do 9 in the first box, 3 in the second and 7 in the last box	... well **if you**, when **you**'re reading books, **if you**'re. Sentences are **like**, well, hard to explain, . something, **You** don't have read them, if there are no full stops, with it **you**'d run out of breath, always have a full stop at the end, well uh,. the idea is to stop **you** running out of breath.
Robert	Well I get hundred, I get **like** one, oh, oh, hundred, I get , I mean I get ten, and I get, no, I get 5 then ten then a hundred.	it's **like** doing sentences, you've got **like**, it's not **like**, "day whe," it's not it's **like** "day", **like** that, it's meant to be "one day" like "One day a boy walked the street" It's **like** that.

The first pair of questions was given to compare the style of mental representations that children might bring to mind when asked to give an arithmetic and non-arithmetic procedure. The second pair compared the mental representation that children might spontaneously bring to mind when presented with words that had been the focus of many arithmetic and non-arithmetic lessons. There is evidence here of a consistency of response in these situations. Comparisons between the mental calculation questions and these last four questions also revealed consistencies.

The four children who successfully manipulated digits to answer "What comes next after 379?" all said a particular three-digit number when asked "What is the first thing that comes into your head when I say 'hundreds, tens and units'?". Pam (above) is typical of this group. The three unsuccessful digit manipulators for the first question gave the following responses to the second of these questions:

Suzy .....Hundreds tens and units mean that the hun, you have to do the units first and the hundreds last.

Geoff .um ten is, the T is for like ten, like 50 that's got 10 in it and, and um, the units are like the like ones, the ones that are loose.

Nikki , there's an H, T, and U.

Each of these captures a surface characteristic of the teacher's representation. Geoff was alluding to the use of linking cubes when groups of ten cubes are joined together and the remaining "ones" are left unlinked (loose).

Seven children gave the answer 380 (for "What comes next after 379?") by counting, adding or implying it was a known fact. None of them gave a particular number as the first thing that came into their heads. Bobby, Joe and Steve gave more general responses (above). The others were:

Joe like it said, well like a pencil written hundred tens and units

Shaun .... just hundreds tens and units.

Paddy . hundred

Elaine . well hundreds tens and units

Whilst these seem not to be very interesting responses it may be of note that these last three were three of only four children who gave the correct answer to all of the calculation questions.

The same high level of consistency between individual children's responses to paired questions was also apparent when comparing their answers to "What is 73 take 20?" and "Tell me how to add 20" . Of the 16 children who gave a response to both questions, four of them counted for the first calculation and said that they would count to add 20. All the other children gave a digit manipulation response to each.

Discussion

The evidence gathered from this small sample confirmed the findings of my previous studies. The linguistic aspects of the children's responses have been reported here but the children were also asked frequently during the interviews if they could "see" anything in their heads. There were very few children who reported any visual imagery. Yet their language indicates that what comes in to their minds when asked to perform a calculation is 'image-like' in that it reproduces aspects of the teacher's representation. The metaphoric language of manipulation, movement and position indicates that many of the children have mental representations formed from their interactions with the teacher's representations. "Take the three hundred off", "added a 3 in front", "300 would come first" are sufficiently different for me to contend that these are the children's own constructions. They are also sufficiently similar to suggest that they have a common source in the teacher's representations. Not all of the children, however, have the same type of mental representation. Many children who were not 'digit-manipulators' used counting-on and counting back as their mental representations for addition and subtraction. These were the main 'types' of mental representations though one child represented addition and subtraction as numbers moving forward or backward.

Children might have been expected to give a description of an image or of the procedure they had just carried out when, after performing a mental calculation, they were asked "What was in your head when you were thinking of that?" Instead, some chose to talk in more general terms of what to do using such language as "you add the tens". Their use of "you" follows the common classroom and peer-group convention of using "you" instead of the more formal "one" to imply that a person in general does such a thing. The use of "the tens" instead of the specific number just used in their calculation also implies that they are recalling a mental representation of a general procedure and using it for this particular calculation. Some children even gave a different example to illustrate their use of the procedure instead of the calculation just they had just done. Their use of "if" frequently indicated an illustrative example both when describing how they had performed their calculation and when asked "Tell me how to ... ". The two 'styles' of mental representation are thus 'particular' and 'general' though many children who gave illustrative examples demonstrated a recognition of generality.

Children in this school have experience of counting objects in Year 1 and concentrate their efforts on addition and subtraction of numbers up to 20. In Year 2 there is little opportunity to handle large numbers of objects and instead they manipulate numerals or representations of the numeral system (Dienes blocks, hundred squares, digits written in columns). The language they use to describe addition and subtraction of multi-digit numbers is the metaphoric language of manipulation of these representations. They have missed primary experiences of handling any large quantities. The use of this language to accompany experience with these external representations helps children construct their understanding of the numeral system but not neccessarily an understanding of quantity. Von Glasersfeld (1995) has noted that the concepts we form in our first language may not be expressible in any other language that we might subsequently learn. For children, their 'first language' of mathematics will shape and form their view of mathematical concepts in a way that may make it difficult for them to express ideas in any different mathematical language learned later. The common problems that arise from inappropriate use of "multiplying makes bigger" and "put a nought on to times by ten" are illustrations of this.

The 'sense that children make' of their teacher's representations seems to involve mental representations that are constrained by the language and experiences with those representations. Their individual constructions re-present those experiences and thus their concepts are formed from the experiences of the representations. The medium has become the message.

Conclusion

There were strong indications in this group of children of a consistency of type of mental representation used when they performed the calculation "...73 take away 20" and when asked "...how to add 20". They described either counting or digit manipulation for each. The children also gave the same style of response when asked the two 'first-thing-in-head' questions and the same style for the two 'tell-me-how' questions. Whilst it is difficult to give an interpretation, it is worth noting that those with different types of mental representations for "after 379" gave qualitatively different responses to 'first-thing-in-head' questions.

The analysis of children's language to categorise their mental representations will allow distinctions to be made between children and to check on the development of the mental representations of individuals. A longitudinal study will be conducted in the same school to explore the differences between individual children in the type of mental representations that they use, how these mental representations are related to the teachers' representations and how the representations develop over time. As children have experience of an extended number system (numbers greater than 100, fractions and decimals) the influence of their early mental representations will be investigated.

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