# DEFINITIONS AND PROCESSES: TWO CONTRASTING APPROACHES TO THE TEACHING AND LEARNING OF LINEAR ALGEBRA 

Alkistis Klapsinou \& Eddie Gray<br>Mathematics Education Research Centre, University of Warwick, Coventry CV4 7AL, alkistis@fcis1.wie.warwick.ac.uk, E.M.Gray@warwick.ac.uk

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#### Abstract

This paper focuses upon the strengths and weaknesses of disparate methods of delivering a Linear Algebra course. Drawing upon observations from within two teaching approaches - one identified as abstraction-to-computation and the other as computation-to-abstraction - the paper reflects upon the quality of assimilation by students who have attended each of these courses. Emerging differences between matched pairs of students suggest that those from the former focus on the definitions of the core concepts of Linear Algebra. In contrast, it is the processes in which these concepts are involved that command the attention of the students in the latter.


## Abstract in Greek










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## Introduction

Linear Algebra is one of the first courses of advanced mathematics at University level. Along with Analysis, it is intended to shift the students' way of thinking from school mathematics towards advanced mathematical thinking. It is probably the first 'real' mathematics course that students have to encounter, since it requires limited mathematical prerequisites, yet its theory is systematically built from the ground up (Tucker, 1993; Hillel \& Sierpinska, 1994). In addition, Linear Algebra brings together methods and insights of geometry and algebra, and its wide range of applications in both the natural physical sciences and modern mathematics make it an essential component of all scientific courses (Harel, 1989; Tucker, 1993). Most importantly though, students have to become familiar with its main themes, such as vector spaces and linear maps, since they are central in the further development of pure mathematical theory (Tucker, 1993).

This paper starts by summarising the existing literature concerning the cognitive obstacles in the field of Linear Algebra and identifies two teaching approaches that aimed to help students overcome these obstacles. By drawing upon specific observations of two similar groups of students it demonstrates how these approaches - one abstract and one concrete - result in qualitatively different learning. Having in mind that the central aim of undergraduate mathematics is to shift the students' way of thinking from high-school mathematics towards Advanced Mathematical Thinking, it suggests that, at least in the case of Linear Algebra, an abstract approach may be the way forward.

## Difficulties within Linear Algebra

During their pre-university courses, U.K. mathematics students will have met some components of the course, such as matrix arithmetic and solution of simultaneous linear equations (AEB GCE Syllabuses, 1999; NEAB GCE A/AS Syllabuses for 1998). This, unfortunately, does not guarantee a smooth transition to the essence of Linear Algebra. On the contrary, as Hillel and Sierpinska (1994) argue, "both the teaching and learning of linear algebra at the university level is almost universally regarded as a frustrating experience" (p. 65).

Some of the reasons for the difficulties faced by students are not confined to the content of Linear Algebra, but are a result of the transition from elementary to advanced mathematics.

The move from elementary to advanced mathematical thinking involves a significant transition: that from describing to defining, from convincing to proving in a logical manner based on those definitions. This transition requires a
cognitive reconstruction which is seen during the university students' initial struggle with formal abstractions as they tackle the first year of university. It is the transition from the coherence of elementary mathematics to the consequence of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions. (Tall, 1991, p. 20)

These abstract entities in the case of Linear Algebra are structures that represent concepts and systems with various properties. All the assumptions are made explicit and all statements are justified by reference to definitions and already proven facts. High school mathematics does not deal with abstract systems and therefore first year undergraduate students are not experienced enough to work with abstract notions and proof-based theories (Harel, 1989; Hillel \& Sierpinska, 1994).

Another difference between high school and university mathematics is how the new knowledge is constructed. In the context of advanced mathematics, generalisations play an important role in the expansion of a conceptual network. Harel \& Tall (1991) distinguish between three types of generalisations; expansive, reconstructive and disjunctive generalisation. To make an expansive generalisation the subject expands the applicability range of an existing schema whilst in a reconstructive generalisation the existing schema has to be reconstructed in order to broaden its applicability range. These are seen in sharp contrast to the notion of disjunctive generalisation, where a new, disjoint schema is constructed to deal with the new concept. In the context of Linear Algebra, the successive generalisations of vector sum and scalar multiples from $\mathbf{R}^{2}$ to $\mathbf{R}^{3}$ to $\mathbf{R}^{n}$ may be seen as an expansive generalisation for the students, since it involves "applying the same techniques to each coordinate in successively broader systems" (Harel \& Tall, 1991, p. 39). The passage from $\mathbf{R}^{n}$ to the abstract concept of a vector space, on the other hand, requires a re-constructive generalisation.

The learner is presented with a name for the concept ("the vector space $V$ ") and some of its properties (the axioms) and - usually guided by an expert - must follow a subtle and difficult process of construction and meaning of $V$ and its properties by deduction from the axioms. This is further complicated in the learner's mind by the fact that the properties to be deduced in V are known to hold $\mathbf{R}^{n}$, causing the problem for the student that, although these properties are "obvious" in the (only) examples (s)he understands, judgement must be suspended on their truth in $V$ until they are shown to follow by deduction from the axioms. (Harel \& Tall, 1991; p. 39)

Linear Algebra was developed to unify, simplify and model already existing problem solutions, rather than to solve new problems (Harel \& Trgalová, 1996). This simplification, essential as it may be for the further development of mathematical theory, it is only visible to the expert who appreciates the advantages of having a generalised theory and can apply the results in many different contexts. From the learner's point of view though, the need for new definitions and theorems is not at all obvious and therefore there is little incentive to acquire the new knowledge (Dorier, Robert, Robinet \& Rogalski, 1998). Most students can solve many problems within a Linear Algebra course without using the relevant theory but by mere application of direct manipulation techniques (Hillel, \& Sierpinska, 1994). As Dorier (1990) argues "there is no problem except a few, far too complicated for students, for which linear algebra is an absolute necessity" (p. 38).

Even though the theory of Linear Algebra is universally applicable, when it comes to solving problems, there is a wide variety of algorithms for any certain task, with the restriction that different algorithms work in different settings. Thus, students are faced with the further difficulty of having to decide which is the most appropriate algorithm to tackle their problem. For example, Carlson (1993) notes that the procedure needed to find a basis for a vector space of row vectors is different to that used to find a basis for a vector space of functions.

An additional disadvantage stemming from the unifying character of Linear Algebra is the variety of representations that students have to get accustomed to. The word 'vector', for example, firstly introduced in the context of concrete $\mathbf{R}^{n}$ subspaces, can mean different things depending on the corresponding vector space. Hillel \& Sierpinska (1994), argue that the initial representation of a vector as a string of numbers becomes shaken when students realise that the representation of a vector depends on the choice of basis.

As Linear Algebra is one of the first undergraduate mathematical courses, students are required - probably for the first time - to deal with abstract concepts instead of numeric manipulations (Carlson, 1993). They have to start "thinking about the objects and operations of algebra not in terms of relations between particular matrices, vectors and operators but in terms of whole structures of such things: vector spaces over fields, algebras, classes of linear operators, which can be transformed, represented in different ways, considered as isomorphic or not, etc." (Hillel \& Sierpinska, 1994, p. 65). An example of how complicated things can get when talking about 'the objects' of Linear Algebra is given by Harel (1989):
...to understand the concept of a vector space of functions, functions must be viewed and handled as mathematical objects in the same way that real numbers are objects. Moreover, students must understand that the two vector-space operations, addition and scalar multiplication, are themselves functions operating on objects which are again functions.(p. 141)

The way of overcoming this confusing situation is by realising that the results in the system of Linear Algebra are derived exclusively from the axioms of a vector space and not from any particular properties that its elements may hold. Unfortunately, the majority of first year university students cannot cope with this realisation (Harel, 1989) and thus focus on the elements of the vector space, instead of its structure.

This particular difficulty is not restricted to the context of Linear Algebra, but is common in almost all areas of mathematics. To understand a new notion in elementary mathematics students have to undergo a cognitive shift incorporating lengthy procedures in mathematical concepts. This conversion of actions or operations into what Piaget (1945) described as "thematised objects of thought or assimilation" (p. 49) was described by the term encapsulation (Dubinsky, 1991).

Cottrill, Dubinsky, Nicholls, Schwingendorf, Thomas \& Vidakovic (1996) formulated the APOS theory, from the acronym of the words action, process, object and schema. Actions are physical or mental transformations of objects to obtain other objects. When these actions become intentional they are characterised as processes which may be encapsulated to form a new object. A coherent collection of these actions, processes and objects, linked in some way, is identified as a schema. A schema can be reflected upon and transformed and thus result in the formation of a new object.

The disadvantage of such an approach in Advanced Mathematics is that the students who are taught in this manner are not given a formal definition of the new object until the end -if then- of this whole learning process. Vinner (1991) argues that "it is hard to train a cognitive system to act against its nature and to force it to consult definitions either when forming a concept image or when working on a cognitive task" (p. 72). This situation can only get worse if the students' concept image has been built through actions and processes, without considering the concept definition.

## Structuring a Linear Algebra course

There seem to be two ways of sequencing the contents of Linear Algebra; the computation-to-abstraction approach and the abstraction-to-computation approach (Harel, 1987). The former suggests that matrix arithmetic and linear systems should precede vector spaces and linear transformations, in order to enable students to develop the language and reasoning needed for understanding the more abstract material. The latter starts with vector spaces and linear maps and then matrices and simultaneous linear equations are treated as applications of the former.

The purpose of this paper is to demonstrate how these two different approaches were applied in two parallel running courses of Linear Algebra and how each approach lead the students' learning in different directions.

Our focus of attention is on students who in the normal course of events appear to have the basic understanding and knowledge to study for a mathematics degree in almost any university within the U.K. However, they are not all following a Mathematics course (M). Some are studying for a Combined Mathematics degree (CM) in, for example, mathematics and physics, or mathematics, operational research, statistics and economics, etc. No matter which degree the students are studying for, they have to take the same core mathematical courses in their first year. One of these is Linear Algebra.

Our fundamental interest evolves around the students' formation of objects within Linear Algebra. As part of this study, pairs of students, one from each group, were matched on the basis of their achievement in a diagnostic test administered during their first week at the University. Five matched pairs of students were identified covering the whole spectrum of achievement. Here we are only reporting on two of these pairs - one representing the high achievers and one representing the low achievers.

## The lectures

The interesting feature of this year's (1998-1999) presentation of the subject matter is that each group had a different lecturer each of whom had contrasting conceptions of the way in which the material should be presented.

The lecturer responsible for the M group sequenced the lectures so that vector spaces preceded linear transformations and matrices. This would fit the abstraction-to-computation approach. In contrast, the lecturer for the CM group sequenced the material according to the computation-to-abstraction approach. The lecturer felt it would be more beneficial to start with already familiar concepts and use them as building blocks for the development of more abstract notions.

Each lecturer's presentation style seemed to be consistent with their view of the sequencing. The M lecturer wrote notes on the board and students copied them. These notes only contained the very essential definitions, axioms and theorems; most of the corollaries and applications were left for the students to prove in their weekly assignments. The CM group, on the other hand, had explicit typed notes that contained everything they needed to know including several examples of each new notion. These examples often preceded the definition of the notion itself. As a result the focus of attention shifted from the concept to the possible application of the concept.

As an example, consider the way in which each lecturer introduced the notion of vector space.

## M group

Vector Space over a field $\mathbf{F}$ is a set $V$ with two operations: + addition, $\bullet$ scalar multiplication.
Addition: exactly the same as for a field, i.e. given two elements $\mathrm{v}, \mathrm{w} \in V$ there is an element $(\mathrm{v}+\mathrm{w}) \in V$ and
A1: $v+w=w+v$
A2: $(\mathrm{v}+\mathrm{w})+\mathrm{z}=\mathrm{v}+(\mathrm{w}+\mathrm{z})$
A3: $\exists 0 \in V$ s.t. $0+\mathrm{v}=\mathrm{v} \forall \mathrm{v} \in V$
A4: Given $v \in V \exists-v \in V$ s.t. $v+(-v)=0$
Scalar multiplication: Given $\mathrm{v} \in V$ and $\alpha \in \mathrm{F}$ there is an element $\alpha \cdot v \in V$ (we often omit the $\bullet$, i.e. $\alpha \mathrm{v} \in V$ ).

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    The elements of \(V\) are called vectors. \(\mathbf{F}\) is called the underlying or ground field. The elements of Fare called
scalars.
Scalar multiplication satisfies: \(\mathrm{S} 1: 1 \cdot \mathrm{v}=\mathrm{v} \forall \mathrm{v} \in V\)
    S2: \((\alpha \beta) v=\alpha(\beta v) \forall \alpha, \beta \in \mathrm{F}, \mathrm{v} \in V\)
Distributive laws \(\mathrm{D} 1:(\alpha+\beta) \mathrm{v}=\alpha \mathrm{v}+\beta \mathrm{v} \forall \alpha, \beta \in \mathrm{F}, \mathrm{v} \in V\)
    D2: \(\alpha(\mathrm{v}+\mathrm{w})=\alpha \mathrm{v}+\alpha \mathrm{w} \forall \alpha \in \mathrm{F}, \mathrm{v}, \mathrm{w} \in V\)
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## CM group

## Vectors

We'll start by summarising the vector operations.
Let x and y be vectors in $\mathbf{R}^{\mathrm{n}}: \mathbf{x}=\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \ldots, \mathrm{x}_{\mathrm{n}}\right)$ and $\mathbf{y}=\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \ldots, \mathrm{y}_{\mathrm{n}}\right)$
The sum of $x$ and $y$ is given by $\mathbf{x}+\mathbf{y}=\left(x_{1}+y_{1}, x_{2}+y_{2}, \ldots, x_{n}+y_{n}\right)$
The scalar multiplication of $x$ by the scalar $b$ is given by $\beta x=\left(\beta x_{1}, \beta x_{2}, \ldots, \beta x_{n}\right)$
The vector multiplication of $x$ and $y$ is given by $\mathbf{x} \cdot \mathbf{y}=x_{1} y_{1}+x_{2} y_{2}+\ldots+x_{n} y_{n}$

## Vector Spaces

Once you have defined vector addition and scalar multiplication by a real number, each of the sets $\mathrm{R}^{\mathrm{n}}$ satisfies the following ten properties.

If $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are vectors in the set $\mathbf{R}^{\mathbf{n}}$, and $\beta$ and $\beta^{\prime}$ are scalars then we have:

P1 Closure under addition
P2 Closure under scalar multiplication
P3 Associative law for addition
P4 Commutative law for addition
P5 Existence of an additive identity
P6 Existence of additive inverses
P7 Associative law for scalar multiplication
P8 Existence of a scalar multiplicative identity
P9 First distributive law
P10 Second distributive law

The vector $\mathbf{x}+\mathbf{y}$ is also in the set.
The vector $\beta \mathbf{x}$ is also in the set.
$\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$
$\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$
$\mathbf{x}+\mathbf{0}=\mathbf{x}$
$\mathbf{x}+(-\mathbf{x})=\mathbf{0}$
$\left(\beta \beta^{\prime}\right) \mathbf{x}=\beta\left(\beta^{\prime} \mathbf{x}\right)$
$1 \mathbf{x}=\mathbf{x}$
$\beta(\mathbf{x}+\mathbf{y})=\beta \mathbf{x}+\beta \mathbf{y}$
$\left(\beta+\beta^{\prime}\right) \mathbf{x}=\beta \mathbf{x}+\beta^{\prime} \mathbf{x}$

Any set that satisfies all the properties P1 to P10 is called a "Vector Space over R". The 'over R' bit comes from the fact that you do scalar multiplication with real numbers (scalars). The big step we make next is to notice that there are sets other the $\mathrm{R}^{n} \mathrm{~s}$ that are vector spaces. In other words there are other sets with addition and scalar multiplication defined on them that satisfy all the properties above. This does not sound too unsettling until you realise that some of these sets have elements that are not even vectors, and yet they are still called vector spaces.

From these two extracts of the notes, it becomes quite apparent that the M group's lecturer follows a more formalistic approach, whereas the CM lecturer demonstrates a method of teaching consistent with the APOS theory (Cotrill et al., 1996), although this was unintentional. Starting with vectors in $\mathrm{R}^{\mathrm{n}}$, objects already familiar to the students, addition and scalar multiplication were defined, initially as actions on these vectors. When these actions were interiorised, along with their properties (the 10 vector space axioms), they became processes, which then were used to form the new object 'vector space'. These notions were later extended to include vector spaces other than $\mathrm{R}^{\mathrm{n}}$, resulting in the schema of a general vector space.

## Interviewing students

The selected pairs of students were interviewed fortnightly during the ten-week period that they followed their courses. The interviews, semi-structured in form, were tape-recorded and focused upon students understanding of key concepts emerging throughout the courses and the way in which these may change as the course progressed. For example, students were asked: "What is a vector space?", "How are the rank of a matrix and the rank of linear transformation associated?"

Additionally, students were presented with multiple choice questions associated with the objects of linear algebra. These were drawn from Jänich (1990) and were chosen because they dealt with the objects of linear algebra rather than the procedural aspects of the course. These questions had shown in trialling that they highlighted the smallest possible misunderstandings and even though they do not specifically ask for definitions or proof of theorems, it was the use or otherwise of these facets which allowed us to consider differences in students' thinking. It was the quality of this thinking that concerned us rather than the individual differences in achievement. Examples of these multiple choice questions will be presented as part of our considerations of the student's responses.

## Students conceptions within Linear Algebra

As we said earlier, we are only going to report here of two matched pairs of students, whom we are going to identify as $\mathrm{M}+$ (mathematics high-achiever), M - (mathematics low achiever), $\mathrm{CM}+$ (combined mathematics high-achiever) and CM- (combined mathematics low-achiever). In particular, we are going to see a few extracts from the transcripts of the
individual interviews and compare the responses and reasons that the same-ability but different-group students gave to the same multiple-choice questions.

We shall show that the $M$ group students focused more on the axioms and the definitions of the objects of Linear Algebra whilst the CM group students used more the properties of these objects and the processes in which they were involved. Additionally the M students were more prone to construct proofs and justify their answers rigorously whereas the CM students relied upon their memory of the things they did within the course (or what they thought they did) when answering the questions.

## Which of the following statements is true?

If $U$ is a subspace of $V$, then $V \backslash U$ is also a subspace of $V$.
$\square \quad$ There exists a subspace $U$ of $V$ for which $V \backslash U$ is also a subspace, but $V \backslash U$ is not a subspace for all subspaces $U$.
$\square \quad$ If $\boldsymbol{U}$ is a subspace of $\boldsymbol{V}$, then $\boldsymbol{V} \backslash \boldsymbol{U}$ is never a subspace of $\boldsymbol{V}$.
$\mathrm{M}+$ : I think yes, it's a complementary subspace. Or is it? No, it's not. No, it's not because if you have $V$ as $\mathbf{R}^{3}$ and $U$ as, let's say, the plane, then what's left is not a subspace. No, that can't be right. ... I have to think about that one. I can't see how you could have a subspace.
[CM + chooses the first]
I: Why is that true?
$\mathrm{CM}+$ : Because we've done a question that there is a complementary subspace, or that every subspace has a complementary subspace.

## I: Have you?

$\mathrm{CM}+$ : I think we did something like that.
This example indicates that even both students were initially wrong in answering the question, $\mathrm{M}+$ was critical about his answer and by thinking it through and constructing a generic example he eventually got to the right result. $\mathrm{CM}+$, on the other hand, confuses the question with something he might have done in another context (e.g. sets and subsets) and thus indicates that he has not constructed the right links between the individual pieces of knowledge. We also have to note that in a previous open question asking what a vector subspace is CM + knew the definition. Even so, instead of using it to answer the multiple-choice question above, he prefers to rely upon something he 'remembers'.

For which of the following objects does the description "linearly dependent" or "linearly independent" make sense?
$\square$ An $n$-tuple $\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ of elements of a vector space
$\square \quad$ An $n$-tuple ( $v_{1}, v_{2}, \ldots, v_{n}$ ) of real vector spaces
$\square$ A linear combination $\lambda_{1} v_{1}+\lambda_{2} v_{2}+\lambda_{n} v_{n}$
$\mathrm{M}+$ : Ahh... The first one. You have a set of vectors being linearly independent. The second doesn't make any sense and the third one is just a vector.
$\mathrm{CM}+$ : Oh, I don't know about this one. I guess the third.
Here we have an example of how familiar with the definition of linear independence is $\mathrm{M}+$ and how this leads him to the correct answer. $\mathrm{CM}+$ on the contrary recalls the formula of how we decide whether a set of vectors is linearly independent and that obstructs him from even reading the question properly. The process here gets in the way of deciding what the object is.

A map $f: V \Pi W$ between vector spaces $V$ and $W$ over $F$ is linear, if
$\square \quad f(\lambda x+\mu y)=\lambda f(x)+\mu f(y)$ for all $x, y \in V, \lambda, \mu \in F$.
$\square \quad f$ satisfies the eight axioms for a vector space.
$f: V \Pi W$ is bijective.
M-: It's the first one. It kind of summarises everything into a nice property.
CM-: If it's linear isn't there a 1-to-1 relationship?
I: Not necessarily.
CM-: Oh...[draws
wouldn't be bijective. Hmm... I think it's the first one... Yeah, I remember doing that. That [ $\lambda x+\mu y$ ] would be an element of the set...

As we can see from this example the low achieving students indicate similar approaches to the questions as the high achievers. When they were asked a question about the defining property of linear transformations M- guided by the axiomatic definition chose the right answer. CM- though seems to have closely associated linear transformations with a Venn diagram (a representation that their lecturer was very fond of), and the properties that go with it, that does not help him very much. On the contrary, this representation becomes the focal point of CM-'s concept image and the definition of linear transformations has become obsolete.

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## $\square \quad$ for each $n$-tuple $\left(v_{1}, \ldots, v_{n}\right)$ in $V$, the $n$-tuple $\left(f\left(v_{1}\right), \ldots, f\left(v_{n}\right)\right)$ is a basis for $W$.

$\mathrm{M}+$ : I think it's the first one. That's saying it's invertible. If it's an isomorphism it has to be invertible, and the nullity is 0 . I can't remember what was exactly the definition and what was the corollary...

## I: The definition was that it's a linear map that it's also bijective.

$\mathrm{M}+$ : I seem to remember that it had to something with being invertible. But it's definitely the first one, because the others don't make sense.
$\mathrm{CM}+$ : Well, if they are isomorphic.
I: It's a circular argument, isn't it?
M-: It's the first one. Well, you can tell it's not the middle one.
I: Well the middle one is just a circular argument.
M-: Did I pick the right answer then?
I: Yes.
M-: Waou!

## I: But do you know why?

M-: Cause I remember doing isomorphisms and you had to be able to do that.
CM-: Well, I can't remember that at all, even what it means.

## I: An isomorphism is a bijective linear map.

CM-: So there's only one thing in the kernel. Hmm...
The example above includes the answers to a question given by all four students and thus allows us to see how the ability range affected the quality of the replies. From all four only M+ demonstrates use of the definition of isomorphism and he actually shows that this definition is not an isolated piece of knowledge; on the contrary he seems to have other concepts - such as that of nullity - strongly associated with it. M- picks the right answer but that is only because she remembers of a process that she has used in the past. As for both the CM students, they seem to have no recollection at all of the definition of isomorphism. Even when the interviewer tells CM- the definition of isomorphism he is unable to construct the links needed to get to the right answer.

In general, we could say that there seems to be a difference between the two groups, especially in the way they use or not use - definitions to reply to questions. The M students are more willing to use the definitions or the theorems they have proved to get to the right answer, whereas the CM students demonstrate a procedural approach to things and their conceptual network seems very unconnected.

As a further proof to this comment we can say that when all 10 students from both groups were asked in their final interview what was the association between the rank of a linear transformation and the rank of the corresponding matrix, all 5 students in the M group knew the answer but only the highest achiever in the CM group was able to respond correctly. Therefore, we can say that the M group students have reconstructed their schema of rank to include both the rank of a matrix and the rank of a linear transformation - reconstructive generalisation (Harel \& Tall, 1991) whilst the CM group students have built two disjoint schemas about the two facets of rank - disjunctive generalisation (ibid.). This is a 'recipe for failure' (ibid.) because

It increases the number of procedures that the individual requires to solve the more general class of problems. It gives the weaker student an additional burden to carry under which he or she is prone to collapse. (p. 38)

## Discussion

The M group lecturer provided the students with just the essential theoretical structures from which they could create their own representations and their own links between the individual pieces of the new knowledge. In their weekly assignments the students had to rigorously construct proofs for several theorems that were not part of the lecture material. High achievers like $\mathrm{M}+$ managed to build a sound knowledge of Linear Algebra. However, as we have seen in the examples above, though students such as M- tended to focus more on the procedural aspects of the problem as the questions became more difficult or detached from their immediate experience, they did appreciate that definitions could provide the basis for abstract entities.

The CM group was provided with an extensive number of examples and multiple representations. Students such as $\mathrm{CM}+$ seemed able to select the representation that was more appropriate to their way of thinking and working. However, though they too could appreciate the strength of abstract entities when dealing with isolated aspects of the course (for example when asked about specific definitions) when questions involved the combination of knowledge from different sections of the course they displayed an unconnected conceptual network. Such a characteristic was also a feature of the thinking associated with CM- but in addition they seemed very confused with all the different examples and representations.

Overall, the quality of students' learning seems to be affected by both the sequencing of the material and the lecturers' style of presentation. It is difficult to discern which is the most influential. On the one hand, we see CM + students who wish to have had the same experience as the M students, but on the other, lower ability CM students valued and appreciated the experience because "it was very similar to how we did things in school". This suggests that our efforts to encourage those behaviours, which are synonymous with Advanced Mathematical Thinking, need to move away from concretised approaches. Introducing abstract entities with rigour may initially frustrate the students, but if
they accept the responsibility for the construction of their own knowledge they may have a better chance of creating a level of thinking appropriate for undergraduate mathematics.

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## Eıбаү $\omega \boldsymbol{\gamma} \boldsymbol{\eta}$




$\Sigma \tau \eta$ M $\varepsilon \gamma \alpha \dot{\lambda} \lambda \eta$ В $\rho \varepsilon \tau \tau \alpha v i ́ \alpha ~ \tau о ~ \alpha v \alpha \lambda \nu \tau \iota \kappa o ́ ~ \pi \rho o ́ \gamma \rho \alpha \mu \mu \alpha ~ \delta ı \alpha \varphi \varepsilon ́ \rho \varepsilon ı ~ \alpha \pi o ́ ~ \sigma \chi о \lambda \varepsilon i ́ o ~ \sigma \varepsilon ~ \sigma \chi о \lambda \varepsilon i ́ o ~ к ı ~ \varepsilon ́ \tau \sigma ı ~ o ı ~ v \varepsilon о \varepsilon є \sigma \alpha \chi \theta \varepsilon ́ v \tau \varepsilon \varsigma ~$











 $\alpha \pi \varepsilon \iota \kappa о$ रí $\varepsilon \varepsilon \omega v$.










[^0]:    A linear map $f: V \rightarrow W$ is called an isomorphism if
    there exists a linear map $g: W \rightarrow V$ with $f g=I d_{W}$ and $g f=I d_{V}$.
    $V$ and $W$ are isomorphic.

